Be sure to *fully justify your response* to each problem.

1. (15 pts) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. Suppose the series $\sum_{n=1}^{\infty}|a_n|$ converges. Show that the power series $\sum_{n=1}^{\infty}a_nx^n$ converges uniformly on the interval $[-1,1]$ to a continuous function.

   Solution: Set $f_n(x)=\sum_{k=1}^{n-1}a_kx^k$ for $n\geq 2$ and $x\in[-1,1]$. Then for $x\in[-1,1]$, we have $|a_kx^k|\leq|a_k|$, so from the Comparison Test it follows that $\sum_{k=1}^{\infty}a_kx^k$ converges for all $x\in[-1,1]$. Thus, $f_n\to f$, and since each f_n is a polynomial and hence continuous, it suffices by 24.3 Theorem to show that f_n converges uniformly to f.

   For $n\geq 2$, set
$$R_n=\sum_{k=1}^{\infty}a_k-\sum_{k=1}^{n-1}a_k$$
   Then since $\displaystyle\sum_{k=1}^{\infty}a_k$ converges, it follows that $\lim_{n\to\infty}R_n=0$.

   We can now show that $f_n\to f$ uniformly as follows. Given $\epsilon>0$, since $\lim R_n=0$, there is an $N\in\mathbb{N}$ such that
$$R_n<\epsilon\quad\text{for } n\geq N.$$
   Then for all $x\in[-1,1]$,
$$\begin{aligned}|f(x)-f_n(x)|&=\left|\sum_{k=n}^{\infty}a_kx^k\right|\\&\leq\sum_{k=n}^{\infty}|a_kx^k|\\&\leq\sum_{k=n}^{\infty}|a_k|=R_n\\&<\epsilon\end{aligned}$$
   This completes the argument that $\sum_{n=1}^{\infty}a_nx^n$ converges uniformly on $[-1,1]$ to a continuous function.

   Here is an alternate solution. Since $|a_kx^k|\leq|a_k|$ for all $k\geq1$ and all $x\in[-1,1]$, and since $\sum_{k=1}^{\infty}|a_k|<\infty$, the Weierstrass M-test (Theorem 25.7) guarantees that $\sum_{k=1}^{\infty}a_kx^k$ converges uniformly on $[-1,1]$. That is, the sequence of partial sums, $f_n(x)=\sum_{k=1}^{n-1}a_kx^k$, converges uniformly to the function $f(x)=\sum_{k=1}^{\infty}a_kx^k$ on $[0,1]$. Since each f_n is a polynomial function, and thus continuous, the limit f must also be continuous.

2. (15 pts) Show that $|\sin(x)-\sin(y)|\leq|x-y|$ for all x,y in $\mathbb{R}$.

   Solution: The inequality holds for $x=y$, so assume $x\neq y$. Since $\sin(x)$ is differentiable for all $x\in\mathbb{R}$ with derivative equal to $\cos(x)$, the Mean Value Theorem applies to show that given $x\neq y$ there is a $c\in\mathbb{R}$ between x and y with
$$\sin(x)-\sin(y)=\cos(c)\cdot(x-y)$$

Then we have

$$\begin{aligned} |\sin(x) - \sin)(y)| &= |\cos(c) \cdot (x-y)| \\ &= |\cos(c)| \cdot |x-y| \\ &\leq |x-y| \quad \text{since } |\sin(c)| \leq 1 \end{aligned}$$

3. (15 pts) Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function. Assume that f is differentiable on $(0,1)$ and that $f(x) + xf'(x) \geq 0$, for all $x \in (0,1)$. Show that $f(x) \geq 0$, for all $x \in [0,1]$.

Solution: Consider the function $g\colon [0,1] \to \mathbb{R}$ given by $g(x) = xf(x)$. Since both $h(x) = x$ and $f(x)$ are differentiable on $(0,1)$, the function g is also differentiable on $(0,1)$, and, by the product rule,

$$g'(x) = f(x) + xf'(x).$$

Using the hypothesis that $f(x)+xf'(x) \geq 0$, we conclude that $g'(x) \geq 0$, for all $x \in (0,1)$, and hence, g is increasing on $(0,1)$.

On the other hand, since both $h(x) = x$ and $f(x)$ are continuous on $[0,1]$, the function g is also continuous on $[0,1]$. Moreover, $g(0) = 0 \cdot f(0) = 0$. Therefore, $\lim_{x\to 0^+} g(x) = 0$. Since g is increasing on $(0,1)$, it follows that $g(x) \geq 0$ for $x \in [0,1)$. Finally, since $\lim_{x\to 1^-} g(x) = g(1)$, we conclude that $g(x) \geq 0$ for all $x \in [0,1]$.

Observe now that $f(x) = g(x)/x$ for $x \in (0,1]$; since $g(x) \geq 0$ and $x > 0$ in that interval, it follows that $f(x) \geq 0$ for $x \in (0,1]$. Finally, since f is continuous on $[0,1]$ and the limit of a non-negative function is nonnegative, we have that

$$f(0) = \lim_{x\to 0^+} f(x) \geq 0.$$

In conclusion, $f(x) \geq 0$ for all $x \in [0,1]$.

4. (15 pts) A *fixed point* of a function f is a value x such that $f(x) = x$. Let $f\colon (a,b) \to \mathbb{R}$ be a differentiable function. Assume that $f'(x) \neq 1$, for all $x \in (a,b)$. Show that f has at most one fixed point.

Solution: Suppose by contradiction that f has at least two fixed points in (a,b). Denoting these two points by x_1 and x_2, we have that $f(x_1) = x_1$ and $f(x_2) = x_2$.

Now, since f is differentiable on (a,b), we may apply the Mean Value Theorem, which guarantees there is an element $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

It follows from our assumptions that the right side of the above equation is equal to

$$\frac{x_2 - x_1}{x_2 - x_1} = 1.$$

Therefore, $f'(c) = 1$. This contradicts the hypothesis that $f'(x) \neq 1$ for all $x \in (a,b)$, and the proof is complete.

5. Let $f(x) = \ln(x)$, let $\displaystyle\sum_{n=0}^{\infty} a_n(x-1)^n$ be the Taylor series for f around $x = 1$, and let $R_n(x)$ be the remainder $R_n(x) = \ln(x) - \displaystyle\sum_{k=0}^{n-1} a_k(x-1)^k$ for $x > 0$.

(a) (10 pts) Using the formula for $R_n(x)$ in §31.3 Taylor's Theorem (p. 250), find an upper bound for $|R_n(x)|$.

Solution: Let $x > 0$, then by 31.3 Taylor's Theorem

$$R_n(x) = \frac{f^{(n)}(y)}{n!}(x-1)^n \quad \text{for some } y \text{ between } x \text{ and } 1. \tag{1}$$

The first step is to find a formula for $f^{(n)}(y)$ where $f(y) = \ln(y)$. We claim that

$$f^{(n)}(y) = (-1)^{n-1}[(n-1)!]\, y^{-n} \tag{2}$$

for all $n \geq 1$. For $n = 1$, we have $f^{(1)}(y) = y^{-1}$ and equation (2) holds in this case. If $f^{(n)}(y) = (-1)^{n-1}[(n-1)!]\, y^{-n}$, then since $dy^{-n}/dy = -ny^{-n-1}$ we have

$$\begin{aligned} f^{(n+1)}(y) &= (-1)^{n-1}[(n-1)!](-n)y^{-n-1} \\ &= (-1)^n[n!]y^{-(n+1)} \end{aligned}$$

and hence equation (2) holds for $n+1$. Thus, by induction we have that equation (2) holds for all $n \geq 1$.

The next step is to assume $x > 0$ is given and then find an upper bound for

$$|R_n(x)| = \left|\frac{f^{(n)}(y)}{n!}(x-1)^n\right| = \left|\frac{(n-1)!}{n!}y^{-n}(x-1)^n\right| = \frac{1}{n}y^{-n}|x-1|^n \tag{3}$$

Note that y^{-n} is a decreasing function of y (its derivative is negative, for example) so over any interval the maximum value is taken on at the left hand endpoint.

We now consider two cases: $x \geq 1$ and $0 < x < 1$.

If $x \geq 1$, then the left hand endpoint of the interval of points between x and 1 is 1, and we have

$$|R_n(x)| \leq \frac{|x-1|^n}{n} \quad \text{for } x > 1 \tag{4}$$

If $0 < x < 1$, then the left hand endpoint of the interval of points between x and 1 is x, and we have

$$\begin{aligned} |R_n(x)| &\leq \frac{1}{n} \cdot \frac{|x-1|^n}{x^n} \\ &\leq \frac{1}{n}\left(\frac{1-x}{x}\right)^n \quad \text{for } 0 < x < 1 \end{aligned} \tag{5}$$

(b) (10 pts) Find all values of $x > 0$ for which it follows from your result in part (a) that $\lim_{n\to+\infty} R_n(x) = 0$.

Solution: The first step is to find those values of r with $r > 0$ for which $\lim r^n/n = 0$. If $0 < r < 1$, then $\lim r^n = 0$ and hence $\lim r^n/n = 0$ by the squeeze lemma. If $r = 1$, then $\lim r^n/n = \lim 1/n = 0$. If $r > 1$, then by l'Hospital's rule we have $\lim r^n/n = \lim \ln(r)r^n = +\infty$. Thus, we have

$$\lim \frac{r^n}{n} = \begin{cases} 0 & \text{if } 0 < r \leq 1 \\ \infty & \text{if } r > 1 \end{cases} \tag{6}$$

If $x > 1$, it folllows from equations (4) and (6) that $\lim |R_n(x)| = 0$ for $1 < x \leq 2$.

If $0 < x < 1$, then it follows from equations (5) and (6) that $\lim |R_n(X)| = 0$ if

$$\begin{aligned} \frac{1-x}{x} &\leq 1 \\ \frac{1}{x} - 1 &\leq 1 \\ \frac{1}{x} &\leq 2 \\ x &\geq \frac{1}{2} \end{aligned}$$

Summary. From Taylor's formula for the remainder $R_n(x)$ given in equation (3) and the upper bounds for the magnitude of the reminder given in equations (4) and (5), we shown have that the limit of the upper bound for the remainder is zero for $\frac{1}{2} \leq x \leq 2$. Hence

$$\lim |R_n(x)| = 0 \quad \text{for } \frac{1}{2} \leq x \leq 2$$

6. Let f be the function defined by

$$f(t) = \begin{cases} 2 & \text{for } t \leq 0 \\ \sin(t) & \text{for } 0 < t \leq \pi/2 \\ t - \pi/2 + 1 & \text{for } t > \pi/2. \end{cases}$$

(a) (4 pts) Determine $F(x) = \displaystyle\int_0^x f(t)\,dt$.

Solution: For $x \leq 0$, we have $F(x) = \int_0^x 2\,dt = 2t\big|_0^x = 2x$
For $0 < x \leq \pi/2$, we have $F(x) = \int_0^x \sin(t)\,dt = -\cos(t)\big|_0^x = 1 - \cos(x)$
For $x > \pi/2$, we have

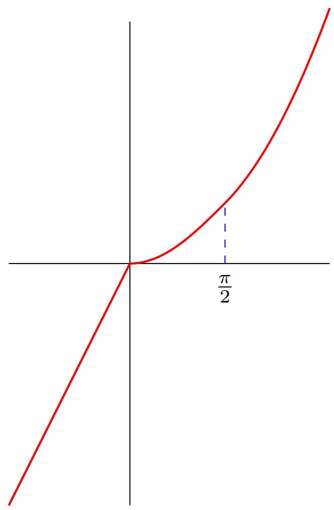
$$\begin{aligned} F(x) &= \int_0^x f(t)\,dt \\ &= \int_0^{\pi/2} f(t)\,dt + \int_{\pi/2}^x f(t)\,dt \\ &= 1 + \int_{\pi/2}^x (t - \pi/2 + 1)\,dt \\ &= 1 + (t^2/2 - (\pi/2)t + t)\big|_{\pi/2}^x \\ &= 1 + x^2/2 - (\pi/2)x + x - [(\pi^2)/8 - (\pi^2)/4 + \pi/2] \\ &= x^2/2 - (\pi/2)x + x + (\pi^2)/8 - \pi/2 + 1 \end{aligned}$$

So we have

$$F(x) = \begin{cases} 2x & \text{for } x \leq 0 \\ 1 - \cos(x) & \text{for } 0 < x \leq \pi/2 \\ x^2/2 - (\pi/2)x + x + (\pi^2)/8 - \pi/2 + 1 & \text{for } x > \pi/2 \end{cases} \tag{7}$$

(b) (4 pts) Sketch the graph of F.

Solution:



(c) (4 pts) At which points, if any, is F not continuous?

Solution: Each of the functions listed in equation (7) is the restriction of a differentiable (and hence continuous) function defined on $\mathbb{R}$. Thus, the only possible points at which F is not continuous are the points with $x = 0$ and $x = \pi/2$. For $x = 0$

$$\begin{aligned}\lim_{x\to 0^-} F(x) &= \lim_{x\to 0^-} 2x = 0\\ \lim_{x\to 0^+} F(x) &= \lim_{x\to 0^+} 1-\cos(x) = 1-\cos(0) = 1-1 = 0\end{aligned}$$

and it follows that $F(x)$ is continuous at $x = 0$ since $\lim_{x\to 0^-} F(x) = \lim_{x\to 0^+} F(x)$. For $x = \pi/2$

$$\begin{aligned}\lim_{x\to \pi/2^-} F(x) &= \lim_{x\to \pi/2^-} 1-\cos(x) = 1-\cos(\pi/2) = 1-0 = 1\\ \lim_{x\to \pi/2^+} F(x) &= \lim_{x\to \pi/2^+} x^2/2-(\pi/2)x+x+(\pi^2)/8-\pi/2+1\\ &= (\pi^2)/8-(\pi^2)/4+\pi/2+(\pi^2)/8-\pi/2+1\\ &= 1\end{aligned}$$

and it follows that $F(x)$ is continuous at $x = \pi/2$ since $\lim_{x\to\pi/2^-} F(x) = \lim_{x\to\pi?2^+} F(x)$.
Summary. $F(x)$ is continuous at all values of x in $\mathbb{R}$.

(d) (4 pts) At which points, if any, is F not differentiable?

Solution: Each of the functions listed in equation (7) is the restriction of a differentiable function defined on $\mathbb{R}$. Thus, the only possible points at which F is not differentiable are the points with $x = 0$ and $x = \pi/2$. For $x = 0$

$$\begin{aligned}
\lim_{x\to 0^-} \frac{F(X) - F(0)}{x - 0} &= \lim_{x\to 0^-} \frac{2x - 0}{x - 0} \\
&= 2 \\
\lim_{x\to 0^+} \frac{F(X) - F(0)}{x - 0} &= \lim_{x\to 0^+} \frac{1 - \cos(x) - 0}{x - 0} \\
&= \lim_{x\to 0^+} \frac{1 - \cos(x)}{x} \\
&= \left.\frac{d(1 - \cos(x))}{dx}\right|_{\text{at } x = 0} \\
&= \sin(0) = 0
\end{aligned}$$

It follows that $F(x)$ is not differentiable at $x = 0$ since

$$\lim_{x\to 0^-} \frac{F(X) - F(0)}{x - 0} \neq \lim_{x\to 0^+} \frac{F(X) - F(0)}{x - 0}$$

For $x = \pi/2$

$$\begin{aligned}
\lim_{x\to \pi/2^-} \frac{F(X) - F(\pi/2)}{x - \pi/2} &= \lim_{x\to \pi/2^-} \frac{(1 - \cos(x)) - 1}{x - \pi/2} \\
&= \lim_{x\to \pi/2^-} \frac{-\cos(x)}{x - \pi/2} \\
&= \lim_{x\to \pi/2^-} \frac{-\cos(x) - (-\cos(\pi/2))}{x - \pi/2} \\
&= \left.\frac{d(-\cos(x))}{dx}\right|_{\text{at } x = \pi/2} \\
&= \sin(\pi/2) = 1
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to \pi/2^+} \frac{F(X) - F(\pi/2)}{x - \pi/2} &= \lim_{x\to \pi/2^+} \frac{\left[x^2/2 - (\pi/2)x + x + (\pi^2)/8 - \pi/2 + 1\right] - 1}{x - \pi/2} \\
&= \left.\frac{d[x^2/2 - (\pi/2)x + x + (\pi^2)/8 - \pi/2 + 1]}{dx}\right|_{\text{at } x = \pi/2} \\
&= x - \pi/2 + 1\big|_{\text{at } x = \pi/2} \\
&= 1
\end{aligned}$$

Since

$$\lim_{x\to \pi/2^-} \frac{F(X) - F(\pi/2)}{x - \pi/2} = 1 = \lim_{x\to \pi/2^+} \frac{F(X) - F(\pi/2)}{x - \pi/2}$$

it follows that F is differentiable at $x = \pi/2$ and $F'(\pi/2) = 1$.

(e) (4 pts) Calculate $F'(x)$ at those points where F is differentiable.

Solution: Each of the functions listed in equation (7) is the restriction of an extended function that is differentiable on $\mathbb{R}$, so at all points other than $x = 0$ and $x = \pi/2$ the derivative of F is the derivative of the corresponding extended function. The computations above show that F is not differentiable at $x = 0$ and F is differentiable at $x = \pi/2$ with $F'(\pi/2) = 1$. The following gives $F'(x)$ at the values of x at which F is differentiable.

$$F'(x) = \begin{cases} 2 & \text{for } x < 0 \\ \sin(x) & \text{for } 0 < x \leq \pi/2 \\ x - \pi/2 + 1 & \text{for } x > \pi/2 \end{cases}$$