For each problem be sure to explain the steps in your argument and fully justify your conclusions.

1. For the power series $\sum_{n=1}^{\infty} \frac{2^n}{(3^{2n})\sqrt[5]{n^3}} x^n$,

   (a) (10 pts) Find the radius of convergence.

   Solution: Use the ratio test

$$\begin{aligned}\lim \left|\frac{a_{n+1}}{a_n}\right| &= \lim \left|\frac{2^{n+1}}{2^n} \cdot \frac{3^{2n}}{3^{2(n+1)}} \cdot \frac{\sqrt[5]{n^3}}{\sqrt[5]{(n+1)^3}} \cdot \frac{x^{n+1}}{x^n}\right| \\ &= \lim \frac{2}{3^2}\left(\frac{n}{n+1}\right)^{5/3} |x| \\ &= \lim \frac{2}{9}\left(\frac{1}{1+1/n}\right)^{5/3} |x| \\ &= \frac{2}{9}|x|.\end{aligned}$$

   By the ratio test, the series converges for $|x| < 9/2$ and diverges for $|x| > 9/2$. Thus, the radius of convergence is $9/2$.

   (b) (10 pts) Find the exact interval of convergence.

   Solution: The endpoints of the interval of convergence are $x = -9/2$ and $x = 9/2$.

   With $x = -9/2$ the series is

$$\sum \frac{2^n}{(3^{2n})\sqrt[5]{n^3}} \cdot \left(\frac{-9}{2}\right)^n = \sum (-1)^n \left(\frac{1}{\sqrt[5]{n^3}}\right) = \sum (-1)^n \left(\frac{1}{n^{3/5}}\right).$$

   This is an alternating series with $a_n = 1/n^{3/5}$, since $a_{n+1} < a_n$, and $\lim a_n = 0$, the series converges by the alternating series test.

   For $x = 9/2$, the series is $\sum 1/n^{3/5}$. This is a p-series with $p = 3/5 \leq 1$, so the series diverges by the p-series test.

   The exact interval of convergence is $[-9/2, 9/2)$.

2. For $n \in \mathbb{N}$, let $f_n(x) = (\sin(x))^n$ and let S equal to the set of real numbers, x, for which $f(x) := \lim_{n\to\infty} f_n(x)$ exists.

   (a) (10 pts) Describe the set S and the function $f(x)$ for $x \in S$.

   Solution: First note that $|\sin(x)| \leq 1$ for all $x \in \mathbb{R}$ and that $|\sin(x)| = 1$ if and only if x is an odd multiple of $\pi/2$. In other words, if we let

$$P = \{x \in \mathbb{R} \mid \exists\, k \in \mathbb{Z} \text{ such that } x = (2k+1)\pi/2\},$$

   then $x \in P$ if and only if $|\sin(x)| = 1$ and $x \notin P$ if and only if $|\sin(x)| < 1$.

   Now recall that, for a fixed $a \in R$ with $|a| < 1$, the sequence $(a^n)_{n\in\mathbb{N}}$ converges, and $\lim_{n\to\infty} a^n = 0$. One way to see this is that the geometric series $\sum_{n=0}^{\infty} a^n$ converges to $1/(1-a)$ if $|a| < 1$, a fact which implies that the terms of the series must converge to 0.

   It follows from the above that $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}(\sin(x))^n = 0$ for all $x \notin P$; that is, $f(x) = 0$ for all $x \notin P$.

If $x \in P$, there are two cases to consider. First assume that $x = (4k+1)\pi/2$ for some $k \in \mathbb{Z}$. Then $\sin(x) = 1$, and so $\sin(x)^n = 1$ for all $n \in \mathbb{N}$. It follows that $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} 1 = 1$, and so $f(x) = 1$ in this case.

Next, assume that $x = (4k-1)\pi/2$ for some $k \in \mathbb{Z}$. Then $\sin(x) = -1$, and so $\sin(x)^n = (-1)^n$ for all $n \in \mathbb{N}$. It follows that $\lim_{n\to\infty} f_n(x)$ does *not* exist in this case (see below).

Putting all this information together, we conclude that the set S of real numbers, x, for which $f(x) := \lim_{n\to\infty} f_n(x)$ exists is equal to

$$S = \{x \in \mathbb{R} \mid \nexists\, k \in \mathbb{Z} \text{ such that } x = (4k-1)\pi/2\}.$$

Moreover, the function $f\colon S \to \mathbb{R}$ is given by

$$f(x) = \begin{cases} 0 & \text{if } x \notin P, \\ 1 & \text{if } x = 4k+1 \text{ for some } k \in \mathbb{Z}. \end{cases}$$

(b) (10 pts) For elements y not in S, give an argument that shows $\lim f_n(y)$ does not exist.

Solution: Let y be a real number such that $y \notin S$. As explained above, such an element must be of the form $y = (4k-1)\pi/2$ for some $k \in \mathbb{Z}$. In this case, $\sin(y) = -1$, and so $\sin(x)^n = (-1)^n$ for all $n \in \mathbb{N}$. We need to show that $\lim_{n\to\infty} f_n(y)$ does not exist, or, equivalently, that the sequence $((-1)^n)_{n\in\mathbb{N}}$ does not converge. We will show that by producing two subsequences that converge to different limits.

- First take the subsequence $((-1)^{2m})_{m\in\mathbb{N}}$. Each term in this subsequence is equal to 1; thus, the subsequence converges to 1.
- Next, take the subsequence $((-1)^{2m-1})_{m\in\mathbb{N}}$. Each term in this subsequence is equal to -1; thus, the subsequence converges to -1.

3. For $n \in \mathbb{N}$, let $f_n\colon [0,\infty) \to \mathbb{R}$ be the function given by

$$f_n(x) = \begin{cases} 1 & \text{if } n-1 \le x \le n, \\ 0 & \text{otherwise.} \end{cases}$$

(a) (10 pts) Show that the sequence (f_n) converges pointwise on $[0,\infty)$ and determine the function $f := \lim_{n\to\infty} f_n$.

Solution: Let $x \in [0,\infty)$. For all $n > x$, then, we have that $x \notin [n-1,n]$, and thus $f_n(x) = 0$. It follows that $\lim_{n\to\infty} f_n(x) = 0$. Since this happens for every $x \ge 0$, we conclude that the sequence (f_n) converges pointwise on $[0,\infty)$. Moreover, the limit function, $f := \lim_{n\to\infty} f_n$, is identically 0; that is, $f = 0$.

(b) (15 pts) Does the sequence (f_n) converge uniformly on $[0,\infty)$?

Solution: No, the sequence (f_n) does not converge uniformly on $[0,\infty)$ to the function $f = 0$. To prove this assertion, we need to show:

$$\exists \epsilon > 0 \text{ such that } \quad \forall N \in \mathbb{N}, \quad \exists n > N \text{ and } \exists x \in S \quad \text{such that} \quad |f_n(x)| \ge \epsilon.$$

Take $\epsilon = 1$, and let $N \in \mathbb{N}$. Then, for any $n > N$ and for any x with $n-1 \le x \le n$, we have that $f_n(x) = 1$, and so $|f_n(x)| \ge \epsilon$.

Alternatively, to show that (f_n) does not converge uniformly to 0 on $[0,\infty)$, it is enough to show that

$$\lim_{n\to\infty} \sup\{|f_n(x)| : x \in [0,\infty)\} > 0$$

In our situation, $\sup\{|f_n(x)| : x \in [0,\infty)\} = 1$, for every $n \in \mathbb{N}$, and so

$$\lim_{n\to\infty} \sup\{|f_n(x)| : x \in [0,\infty)\} = 1 > 0.$$

4. Let $f_n(x) = \dfrac{2n - \cos^2(3x)}{5n + \sin(x)}$.

(a) (10 pts) Show that (f_n) converges uniformly on $\mathbb{R}$. *Hint*: First decide what the limit function is and then show that convergence is uniform.

Solution: For all $x \in \mathbb{R}$, we have

$$0 \le \frac{\cos^2(3x)}{n} \le \frac{1}{n} \quad \text{and} \quad \frac{-1}{n} \le \frac{\sin(x)}{n} \le \frac{1}{n}$$

and it follows from the squeeze lemma that $\lim_{n\to\infty} \cos^2(3x)/n = \lim_{n\to\infty} \sin(x)/n = 0$ for all $x \in \mathbb{R}$. Hence for all $x \in \mathbb{R}$ we have

$$\lim_{n\to\infty} \frac{2n - \cos^2(3x)}{5n + \sin(x)} = \lim \frac{2 - \cos^2(3x)/n}{5 + \sin(x)/n} = \frac{2-0}{5+0} = 2/5$$

Thus $f_n \to f$ pointwise, where $f(x) = 2/5$ for all $x \in \mathbb{R}$.

The first step to show that $f_n \to f$ uniformly on $\mathbb{R}$, is to find an upper bound for $|f_n(x) - f(x)|$ that is independent of x. Consider the following

$$\begin{aligned}
|f_n(x) - f(x)| &= \left|\frac{2 - \cos^2(3x)/n}{5 + \sin(x)/n} - \frac{2}{5}\right| \\
&= \left|\frac{10 - 5\cos^2(3x)/n - 10 - 2\sin(x)/n}{5(5 + \sin(x)/n)}\right| \\
&= \left|\frac{-5\cos^2(3x)/n - 2\sin(x)/n}{5(5 + \sin(x)/n)}\right| \\
&\le \frac{5/n + 2/n}{5(5)} = \frac{7}{25}\cdot\frac{1}{n},
\end{aligned} \tag{1}$$

where the last line follows from the line just before it by applying the triangle inequality and using the inequalities $|\cos^2(3x)| \le 1, |\sin(x)| \le 1$, and $5(5 + 1/n) > 25$.

We can now prove that $f_n \to f$ uniformly as follows. Given $\epsilon > 0$, choose $N \in \mathbb{N}$ with $N > \dfrac{7}{25}\cdot\dfrac{1}{\epsilon}$. Then for $n \ge N$, we have $\dfrac{7}{25}\cdot\dfrac{1}{n} < \epsilon$, and hence, using equation (1) we have

$$|f_n(x) - f(x)| < \frac{7}{25}\cdot\frac{1}{n} < \epsilon \quad \text{for all } n \ge N \text{ and all } x \in \mathbb{R}.$$

This completes the proof that $f_n \to f$ uniformly.

(b) (10 pts) Using your result in part (a) and results in the text, determine $\lim_{n\to\infty} \int_a^b f_n(x)\,dx$ for $a < b$. B sure to cite any results you use to justify your answer.

Solution: Since $f_n \to f$ uniformly and each f_n is a continuous function, it follows (see 25.2 Theorem in the text) that for $a < b$

$$\lim_{n\to\infty} \int_a^b \frac{2n - \cos^2(3x)}{5n + \sin(x)}\,dx = \int_a^b \frac{2}{5}\,dx = \frac{2}{5}(b-a).$$

5. (15 pts) For $n \in \mathbb{N}$, let $f_n\colon [0,1] \to \mathbb{R}$ be the function given by $f_n(x) = \sum_{k=0}^{n} \frac{x^k}{2^k}$. Show that the sequence (f_n) is uniformly Cauchy on $[0,1]$.

Solution: For this problem we use the following property of geometric series in Example 1 on page 96 of the text. The series

$$a + ar + ar^2 + \cdots + ar^n + \cdots = \frac{a}{1-r} \quad \text{if } |r| < 1 \tag{2}$$

To see that $f_n(x)$ is uniformly Cauchy on $[0,1]$, the first step is to find an upper bound for $|f_n(x) - f_m(x)|$ that is independent of x. Let $x \in [0,1]$ and let $m, n \in \mathbb{N}$ with $m > n$, then

$$|f_n(x) - f_n(x)| = \left| \sum_{k=0}^{m} \frac{x^k}{2^k} - \sum_{k=0}^{n} \frac{x^k}{2^k} \right|$$

$$= \left| \sum_{k=n+1}^{m} \frac{x^k}{2^k} \right| \tag{a}$$

$$= \left(\frac{x}{2}\right)^{n+1} + \left(\frac{x}{2}\right)^{n+2} + \cdots + \left(\frac{x}{2}\right)^{m} \tag{b}$$

$$< \left(\frac{x}{2}\right)^{n+1} + \left(\frac{x}{2}\right)^{n+2} + \cdots + \left(\frac{x}{2}\right)^{m} + \cdots \tag{c}$$

$$= \frac{(x/2)^{n+1}}{1 - (x/2)} \tag{d}$$

$$\leq 2\left(\frac{x^{n+1}}{2^{n+1}}\right) \tag{e}$$

$$\leq \frac{1}{2^n}, \tag{f}$$

where line (b) follows from line (a) and line (c) follows from line (b) since $x \geq 0$, line (d) follows from line (c) using (2) given that since $x \in [0,1]$, we have $|r| = |x/2| < 1$, line (e) folllows from line (d) since $2 > 1 - x/2$ for $x \in [0,1]$, and line (f) follows from line (e) since $x \in [0,1]$ implies that $x^{n+1} \leq 1$. This shows that

$$|f_n(x) - f_m(x)| < \frac{1}{2^n} \quad \text{for } m \geq n \text{ and all } x \in [0,1]. \tag{3}$$

The proof that $f_n \to f$ uniformly on $[0,1]$ now proceeds as follows. Given any $\epsilon > 0$, since $\lim 1/2^n = 0$, it follows that there is an $N \in \mathbb{N}$ such that $1/2^n < \epsilon$ for all $n \geq N$. Then for $m \geq n \geq N$ we have from equation (3) that

$$|f_n(x) - f_m(x)| < \frac{1}{2^n} < \epsilon$$

and the proof that $f_n \to f$ uniformly is complete.