Real Analysis, MATH 3150 **Final Exam** **Fall 2022**

For this exam you are allowed one two-sided page of notes on a standard, 8 $\frac{1}{2}$ by 11 inches, piece of paper. No additional notes or scratch paper are allowed. You may use the blank, unnumbered, pages on the back of each numbered page for your work if needed. If you do this, be sure to note on the numbered page where the reader should look for the continuation of your work on the problem.
Cellphones and laptops must be turned off and placed on the floor.
For credit you need to fully justify your response to each question. You can cite results in the text by indicating the result—for example, *since every bounded sequence contains a convergent subsequence, it follows that* ......

1. Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x \sin\left(\dfrac{1}{2\pi x}\right), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

(a) (4 pts) Show that f is continuous.

Solution: The function f is differentiable (and thus continuous) at any point $x \neq 0$, since there it is the product of the continuous function x and the composite of the continuous functions $\sin(x)$ and $1/(2\pi x)$.

At $x = 0$, we have

$$\lim_{x\to 0} |f(x)| = \lim_{x\to 0} \left| x \sin\left(\frac{1}{2\pi x}\right)\right| \leq \lim_{x\to 0} |x| = 0,$$

since the sine function is bounded in absolute value by 1. It follows that

$$\lim_{x\to 0} f(x) = 0 = f(0),$$

thus showing that f is also continuous at $x = 0$.

(b) (4 pts) Show that the restriction of f to the interval $[-1, 1]$ is uniformly continuous.

Solution: Since f is continuous on the closed interval $[-1, 1]$, it is uniformly continuous on that interval.

(c) (4 pts) Show that f is *not* differentiable at $x = 0$.

Solution: The limit of the Newton quotient,

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \sin\left(\frac{1}{2\pi x}\right)$$

does not exist. Thus, the function f is not differentiable at $x = 0$.

2. Consider the function $f\colon [0,1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 1, & \text{if } x = 1/3, \\ 2, & \text{if } x = 2/3, \\ 0, & \text{otherwise.} \end{cases}$$

(a) (6 pts) Compute the lower and upper integrals, $L(f) = \underline{\int_0^1} f(x)dx$ and $U(f) = \overline{\int_0^1} f(x)dx$.

Solution: For each $n > 2$, consider the partition of $[0,1]$ given by

$$P_n = \left\{0 < \frac{1}{3} - \frac{1}{3n} < \frac{1}{3} + \frac{1}{3n} < \frac{2}{3} - \frac{1}{3n} < \frac{2}{3} + \frac{1}{3n} < 1\right\}.$$

Then $L(f, P_n) = 0$ and

$$U(f, P_n) = \frac{2}{3n} \cdot 1 + \frac{2}{3n} \cdot 2 = \frac{2}{3n} \cdot 3 = \frac{2}{n}.$$

Therefore, $\lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} U(f, P_n) = 0$.
In general, for any sequence of partitions P_n, we have

$$L(f, P_n) \leq L(f) \leq U(f) \leq U(f, P_n).$$

For f and P_n in this problem, it follows from the Squeeze Lemma that

$$L(f) = U(f) = 0.$$

(b) (2 pts) Show that f is integrable, and compute $\int_0^1 f(x)dx$.

Solution: Since $L(f) = U(f) = 0$, the function f is integrable on the interval $[0,1]$ and $\int_0^1 f(x)dx = 0$.

3. Consider the function $f\colon [1,\infty) \to \mathbb{R}$ given by

$$f(x) = \int_1^{\sqrt{x}} e^{t^2}\,dt.$$

(a) (2 pts) What is $f(1)$?

Solution: $f(1) = \int_1^1 e^{t^2}\,dt = 0$, since the interval of integration has width 0.

(b) (5 pts) Show that f is differentiable. What is its derivative?

Solution: The function $t \mapsto e^{t^2}$ is continuous; thus, by the Fundamental Theorem of Calculus, the function $f(x) = \int_1^{\sqrt{x}} e^{t^2}\,dt$ is differentiable, with derivative

$$f'(x) = e^{(\sqrt{x})^2} \cdot (\sqrt{x})' = \frac{e^x}{2\sqrt{x}}.$$

(c) (3 pts) What is $f'(4)$?

Solution: $f'(4) = \dfrac{e^4}{2\sqrt{4}} = \dfrac{e^4}{4}$.

4. This problem concerns the integral $\int_0^1 x\,dx$.

(a) (5 pts) For a fixed n, let P_n be the partition $\{0 = x_0 < \cdots < x_n = 1\}$ of $[0,1]$ into n equal intervals, so that $x_i = \frac{i}{n}$. Compute the upper and lower sums $U(x, P_n)$ and $L(x, P_n)$. (You may use the formula $1 + 2 + \cdots + k = k(k+1)/2$.)

Solution: $f(x) = x$ is a increasing function so on any interval the maximum value is taken on at the right hand endpoint and the minimum value is taken on at the left hand endpoint. This gives

$$\begin{aligned} U(x, P_n) &= \sum_{i=1}^{n} f((i)/n)(1/n) = \sum_{i=1}^{n} (i) \cdot (1/n^2) \\ &= (1 + 2 + 3 + \cdots + n) \cdot (1/n^2) = \frac{n(n+1)}{2} \cdot \frac{1}{n^2} \\ &= \frac{n^2 + n}{2n^2} = \frac{1 + 1/n}{2} \end{aligned}$$

and

$$\begin{aligned} L(x, P_n) &= \sum_{i=1}^{n} f((i-1)/n)(1/n) = \sum_{i=1}^{n} (i-1) \cdot (1/n^2) \\ &= (0 + 1 + 2 + 3 + \cdots + (n-1)) \cdot (1/n^2) = \frac{(n-1)(n)}{2} \cdot \frac{1}{n^2} \\ &= \frac{n^2 - n}{2n^2} = \frac{1 - 1/n}{2} \end{aligned}$$

(b) (4 pts) Show that $\lim_{n\to\infty} U(x, P_n) = \lim_{n\to\infty} L(x, P_n) = \frac{1}{2}$.

Solution: From part (a) we have

$$\lim U(f, P_n) = \lim \frac{1 + 1/n}{2} = \frac{1}{2} \quad \text{and} \quad \lim L(f, P_n) = \lim \frac{1 - 1/n}{2} = \frac{1}{2}$$

(c) (3 pts) Using the results of parts (a) and (b), deduce that $f(x) = x$ is integrable on $[0,1]$ and $\int_0^1 x\,dx = \frac{1}{2}$.

Solution: In general, for any sequence of partitions P_n, we have

$$L(f, P_n) \le L(f) \le U(f) \le U(f, P_n)$$

For f and P_n in this problem, it follows from the result in part (b) and the Squeeze Lemma that $L(f) = U(f) = 1/2$. Hence $f(x) = x$ is integrable and $\int_0^1 x\,dx = \frac{1}{2}$.

5. (a) (5 pts) Fix $a > 0$ and consider the power series $f_a(x) = \sum_{n\geq 1} \frac{1}{n}(ax)^n$. Determine its radius of convergence R.

Solution: For the power series $f_a(x) = \sum_{n=1}^{\infty} \frac{a^n}{n} x^n$ we have (by the Root Test)

$$\beta = \limsup \left| \frac{a^n}{n} \right|^{1/n} = a \lim_{n\to\infty} (1/n)^{1/n} = a \lim_{x\to 0} x^x.$$

Now note that $x^x = e^{x \ln x} = e^{\frac{\ln x}{1/x}}$. By l'Hôpital's rule,

$$\lim_{x\to 0} \frac{\ln x}{1/x} = \lim_{x\to 0} \frac{1/x}{-1/x^2} = \lim_{x\to 0}(-x) = 0.$$

Hence, $\lim_{x\to 0} x^x = e^0 = 1$, and thus $\beta = a$. We conclude that

$$R = \frac{1}{\beta} = \frac{1}{a}.$$

Alternatively, we may use the Ratio Test to compute the radius of convergence:

$$\beta = \lim_{n\to\infty} \left| \frac{\dfrac{a^{n+1}}{n+1}}{\dfrac{a^n}{n}} \right| = a \lim_{n\to\infty} \frac{n}{n+1} = a,$$

and so, once again, $R = 1/a$.

(b) (5 pts) Compute $f_a'(x)$ on $(-R, R)$, and identify this with a known function in closed form. Using the fundamental theorem of calculus, find an explicit expression for $f_a(x)$.

Solution: For $x \in (-1/a, 1/a)$ we have

$$f_a'(x) = \sum_{n=1}^{\infty} \frac{a^n}{n} \cdot n x^{n-1} = \sum_{n=1}^{\infty} a^n x^{n-1} = a \sum_{k=0}^{\infty} (ax)^k = a \frac{1}{1-ax},$$

where we used the formula for the sum of a geometric series, $\sum_{k=0}^{\infty} r^k = 1/(1-r)$, with ratio $|r| = |ax| < 1$.

Integrating the resulting function, we obtain

$$f_a(x) = a \int \frac{1}{1-ax} dx = -a \ln(1-ax)/a = -\ln(1-ax).$$

(c) (4 pts) Evaluate the series $\sum_{n=1}^{\infty} \frac{1}{n\, 2^n}$.

Solution: This series is of the form $f_a(x)$, with $a = 1$ and $x = 1/2$. Hence,

$$\sum_{n=1}^{\infty} \frac{1}{n\, 2^n} = -\ln(1 - 1 \cdot 1/2) = -\ln(1/2) = \ln(2).$$

6. Consider the sequence of functions

$$f_n(x) = \frac{x}{1+nx^2} \qquad \text{for } x \in \mathbb{R},\ n \in \mathbb{N}.$$

(a) (5 pts) Compute the derivative $f_n'(x)$. Then find the pointwise limit $g(x)$ of the sequence of derivatives $f_n'(x)$ as $n \to \infty$.

Solution: The derivative of $f_n(x)$ is given by

$$f_n'(x) = \frac{1-nx^2}{(1+nx^2)^2}$$

Divding the numerator and denominator each by n^2 gives

$$f_n'(x) = \frac{1/n^2 - x^2/n}{(1/n + x^2)^2}$$

If $x \neq 0$, then $\lim_{n\to\infty} f_n'(x) = 0/x^4 = 0$.
Since $f_n'(0) = 1$, we have $\lim_{n\to\infty} f_n'(0) = 1$.

(b) (5 pts) Does the sequence of derivatives $f_n'(x)$ converge uniformly? Why, or why not?

Solution: From part (a) we have

$$\lim_{n\to\infty} f_n'(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

If the sequence of derivatives $f_n'(x)$ converged uniformly, then the function $\lim_{n\to\infty} f_n'(x)$ would be continuous. Since $\lim_{n\to\infty} f_n'(x)$ is not continuous at $x = 0$, it follows that the sequence of derivatives $f_n'(x)$ does not converge uniformly.

7. Let $g(x) = f(x) + 2x$, where $f\colon [0,1] \to \mathbb{R}$ is a differentiable function which satisfies $f(0) = f(1)$.

(a) (4 pts) Show that there exists a point $c \in [0,1]$ such that $g'(c) = 2$.

Solution: $g'(x) = f'(x) + 2$ so the condition that $g'(c) = 2$ is equivalent to the condition that $f'(c) = 0$. Since $f(0) = f(1)$, it follows from Rolle's Theorem that there is point $c \in (0,1)$ with $f'(c) = 0$, and it then follows that $g'(c) = 2$.

(b) (4 pts) Find explicitly such a point $c \in [0,1]$ for $f(x) = x(x-1)$.

Solution: $g(x) = x(x-1) + 2x = x^2 - x + 2x = x^2 + x$, so $g'(x) = 2x + 1$, and we have $g'(c) = 2$ for $c = 1/2$.

8. For the series
$$\sum_{n=1}^{\infty}(-1)^n\,\frac{x^3+n}{n^3}$$

(a) (5 pts) Show that the series converges uniformly on any bounded interval $[0,M]$ for $M>0$. (*Hint: Use the M-Test.*)

Solution: Fix $M>0$. For $x\in[0,M]$, we have:
$$\left|(-1)^n\,\frac{x^3+n}{n^3}\right| = \frac{x^3+n}{n^3} \le \frac{M^3+n}{n^3}.$$
Moreover,
$$\sum_{n=1}^{\infty}\frac{M^3+n}{n^3} = M^3\sum_{n=1}^{\infty}\frac{1}{n^3}+\sum_{n=1}^{\infty}\frac{1}{n^2}<\infty,$$
since the latter series are p-series, with $p=3$ and $p=2$, respectively, and thus both are convergent (since in both cases $p>1$).

Therefore, by the Weierstrass M-test, the given series converges uniformly on $[0,M]$.

(b) (5 pts) Show that the series does not converge uniformly on $[0,+\infty)$. (*Hint: Use Cauchy's criterion.*)

Solution: Let $s_n(x)=\sum_{k=1}^{n}(-1)^k\,\frac{x^3+k}{k^3}$ be the partial sums of the given series. By the Cauchy criterion for uniform convergence (with $\varepsilon=1$), it is enough to show: For every $N\in\mathbb{N}$, there exists an $x\in\mathbb{R}$ and integers $n\ge m>N$ such that $|s_n(x)-s_m(x)|\ge 1$.

Let us take $m=N+1$ and $n=N+2$; then
$$|s_n(x)-s_{n-1}(x)| = \frac{x^3+n}{n^3} \ge (x/n)^3 = (x/(N+2))^3.$$
Thus, if we let x be a real number such that $x>N+2$ (which we can do, by the Archimedean principle), we have that
$$|s_n(x)-s_{n-1}(x)| \ge (x/(N+2))^3 > ((N+2)/(N+2))^3 = 1,$$
and we are done.

9. Let $f \colon [2,3] \to \mathbb{R}$ be a function, continuous on $[2,3]$, and differentiable on $(2,3)$. Suppose that $f(2) = 6$ and $f(3) = 9$.

(a) (5 pts) Show that, for some point $x_0 \in (2,3)$, the tangent line to the graph of f at x_0 passes through the origin.

Solution: The slope of the line through the origin and the point $(x, f(x))$ on the graph of f is $f(x)/x$. Thus, the condition that the tangent line at the point $(x_0, f(x_0))$ passes though the orgin is

$$\text{slope of tangent} = \text{slope of line through } (0,0) \text{ and the point on the graph}$$

$$f'(x_0) = \frac{f(x_0)}{x_0} \tag{1}$$

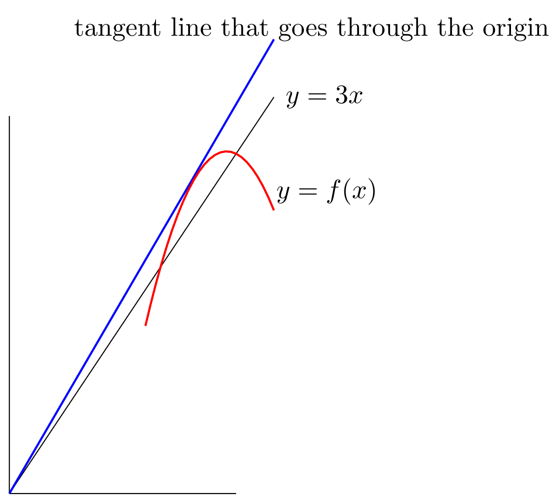
Set $q(x) = \frac{f(x)}{x}$ for $x \in [2,3]$. Then

$$q'(x) = \frac{1}{x}\left(f'(x) - \frac{f(x)}{x}\right)$$

Since $q(2) = q(3) = 3$, it follows from Rolle's Theorem that there is a point x_0 between 2 and 3 with $q'(x_0) = 0$. From the formula above for the derivative of q it then follows that equation (1) holds at x_0, and hence, the argument is complete.

(b) (5 pts) Illustrate your result in part (a) with a sketch.

Solution:



10. (6 pts) Suppose the function $f\colon \mathbb{R} \to \mathbb{R}$ satisfies

$$\big|f(x) - f(y)\big| \le C|x-y|^2 \qquad \text{for all } x, y \in \mathbb{R},$$

for some $C > 0$. Show that f must be constant. (*Hint: First show that f is differentiable.*)

Solution: Suppose $\epsilon > 0$, and $0 < \delta < C/\epsilon$. Then

$$\left|\frac{f(x) - f(y)}{x-y}\right| < C|x-y| < \epsilon \quad \text{for } |x-y| < \delta \tag{2}$$

From the inequality (2), it follows that f is differentiable and $f'(x) = 0$ for all x.

From the Mean Value Theorem it then follows that for $x \neq y$ there is a point c between x and y with

$$f(x) - f(y) = f'(c)(x-y)$$

Since $f'(c) = 0$ we have that $f(x) = f(y)$ and the argument is complete.